NECESSARY AND SUFFICIENT CONDITIONS FOR THE GHS INEQUALITY WITH APPLICATIONS TO ANALYSIS AND PROBABILITY

BY

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ABSTRACT. The GHS inequality is an important tool in the study of the Ising model of ferromagnetism (a model in equilibrium statistical mechanics) and in Euclidean quantum field theory. This paper derives necessary and sufficient conditions on an Ising spin system for the GHS inequality to be valid. Applications to convexity-preserving properties of certain differential equations and diffusion processes are given.

I. **Main results.** In this paper we extend earlier results on the Griffiths-Hurst-Sherman (GHS) inequality [EMN] and apply them to derive convexity-preserving properties of certain differential equations and diffusion processes. The GHS inequality is an important tool in equilibrium statistical mechanics and Euclidean quantum field theory. (For a discussion of physical applications, we refer the reader to the introduction and references of [EMN].) The inequality will be discussed in §II of this paper in order to highlight the connection between statistical mechanics and differential equations. Theorem 2.4 of §II gives necessary and sufficient conditions for the GHS inequality to be valid and is the main technical tool used in deriving the results of the present section. A self-contained proof of this theorem is given in §IV. The theorems of §I are proved in §III with the aid of the results in §II.

In order to state our results, we define three classes of real-valued functions:

$$\mathcal{V} = \left\{ V \mid V: \mathbf{R} \to \mathbf{R} \text{ is even, continuous, and } \lim_{|x| \to \infty} V(x) = \infty \right\},$$

$$\mathcal{V}_c = \left\{ V \mid V \in \mathcal{V}, V(x) = \text{const} + \int_0^x G(y) \, dy \right.$$

(1.1)

$$\left. \text{with } G(0) = 0 \text{ and } G \text{ convex on } [0,\infty) \right\},$$

$$\exp(-\mathcal{V}_c) = \{ f \mid f = \exp(-V) \text{ for some } V \in \mathcal{V}_c \}.$$

We will say that a finite measure ρ on $\mathbf{R}$ belongs to $\exp(-\mathcal{V}_c)$ if ρ is absolutely continuous with respect to Lebesgue measure dx and $d\rho/dx \in \exp(-\mathcal{V}_c)$. Note that the class $\mathcal{V}_c$ in (1.1) can be characterized in various ways (V' denotes dV/dx):

$$\mathcal{V}_c = \left\{ V \mid V \in \mathcal{V}, V, \text{ is differentiable except at } x = 0, \right.$$

(1.2)

$$\left. V' \text{ is convex on } (0,\infty), \text{ and } \lim_{x \to 0^+} V'(x) \leqslant 0 \right\},$$

$$\mathcal{V}_c = \left\{ V \mid V \in \mathcal{V} \text{ and } V(x) = V_1(x) - \gamma|x| \text{ for some } \gamma > 0, \right.$$

(1.3)

$$\left. \text{where } V_1 \in C^1(\mathbf{R}) \text{ with } V_1 \text{ convex on } [0,\infty) \right\}.$$

THEOREM 1.1. *Given* $f \in \exp(-\mathcal{V}_c)$ *and* $V \equiv 0$ *or* $V \in \mathcal{V}_c$, *we denote by* $u(t, x)$ *the unique* $L^2(\mathbf{R}; dx)$ *solution of the parabolic partial differential equation*

(1.4) $$\frac{\partial u}{\partial t} = -H_V u \quad (t > 0, x \in \mathbf{R}), \qquad u(t, \cdot) \to f \quad \text{as } t \to 0^+,$$

where H_V *is the differential operator,*

(1.5) $$H_V = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) \cdot .$$

Then for all $t > 0$, $u(t, \cdot) \in \exp(-\mathcal{V}_c)$.

For $V \in \mathcal{V}$, there exists a basis $\{ \Omega_i; i = 0,1,2 \dots \}$ of $L^2(\mathbf{R}; dx)$ consisting of eigenfunctions of H_V:

$$H_V \Omega_i = E_i \Omega_i, \qquad i = 0,1,2, \dots,$$

where $\Omega_0 > 0$ and $E_0 < E_1 < E_2 < \cdots$ [Ti, Chapter 5], [S1, Theorem II.1.5]. H_V may be thought of as the Hamiltonian of a one-dimensional quantum mechanical system, and our next result concerns the ground state Ω_0.

THEOREM 1.2. *If* $V \in \mathcal{V}_c$, *then* $\Omega_0 \in \exp(-\mathcal{V}_c)$.

By way of comparison, we mention the result of [BL, Theorem 6.1]: If $V \in \mathcal{V}$ is convex on $\mathbf{R}$, then $\Omega_0 = \exp(-G)$ with G convex on $\mathbf{R}$.

The GHS inequality has also been applied to properties of the eigenvalues. It is known, for example [S2, p. 335], [S3], [EMN], that when $V \in \mathcal{V}_c$,

$E_2 - E_1 > E_1 - E_0$ (with equality for $V = \text{const } x^2$), and that $(d/da)(E_1(a) - E_0(a)) > 0$ for $a > 0$, where $E_0(a) < E_1(a)$ denote the lowest two eigenvalues of $-\frac{1}{2}(d^2/dx^2) + V(x) - ax$.

To apply the GHS inequality to probability theory, we associate to $H_V, V \in \mathcal{V}_c$, an operator

$$L_V = \frac{1}{2} \frac{\partial^2}{\partial x^2} + F(x) \frac{\partial}{\partial x},$$

where

(1.6) $$F(x) = (d/dx) \ln \Omega_0(x).$$

This definition is legitimate because $\Omega_0 > 0$ and $\Omega_0 \in C^{2+\alpha}(\mathbf{R})$, any $0 < \alpha < 1$ [**BJS**, p. 136]. The operator L_V is the infinitesimal generator of a unique[3] (up to choice of initial distribution) one-dimensional diffusion process $Y_V(t)$, $t > 0$, whose invariant distribution is $(\Omega_0(x))^2 dx / \int (\Omega_0(x))^2 dx$. For example, when $V(x) = x^2$, $Y_V(t)$ is the Ornstein-Uhlenbeck velocity process. We follow the standard practice of writing E_x to denote expectations with respect to the process $Y_V(t)$ with $Y_V(0) = x$. The next theorem gives convexity-preserving properties for the "backward" and "forward" diffusion equations associated with L_V.

THEOREM 1.3. (a) *Given* $f \in C(\mathbf{R})$, *define*

(1.7) $$h(t,x) = E_x f(Y_V(t)),$$

which is a solution of the Cauchy problem

$$\partial h/\partial t = L_V h, \qquad h(t,\cdot) \to f \quad as \ t \to 0^+.$$

If $\Omega_0 f \in \exp(-\mathcal{V}_c)$, *then for all* $t > 0$, $\Omega_0(\cdot)h(t,\cdot) \in \exp(-\mathcal{V}_c)$.

(b) *Let* $\rho_t(dx)$ *denote the probability distribution of* $Y_V(t)$ *for* $t > 0$. *If* $\Omega_0^{-1}\rho_0(dx) \in \exp(-\mathcal{V}_c)$, *then for all* $t > 0$, $\Omega_0^{-1}\rho_t(dx) \in \exp(-\mathcal{V}_c)$.

(c) *Both results* (a) *and* (b) *and their converses hold for the Brownian motion process* ($V \equiv 0$) *if we formally set* $\Omega_0 \equiv 1$.

REMARK 1.4. There are several simple extensions of these theorems which can be obtained by utilizing more fully the results of Theorem 2.4 and Proposition 2.7. First, all conclusions remain valid when f in Theorem 1.1 or $\Omega_0 f$ in Theorem 1.3 has the form given by the right-hand side of (2.7) with $I < \infty$. The same is true if $\Omega_0^{-1}\rho_0$ is assumed only to be in $\mathcal{G}$ (see Theorem 2.2). Second, all the results of this section extend in a natural way to the analogous differential equations on a finite interval $(-I, I)$ with Dirichlet boundary conditions and to their related diffusion processes.

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[3]It suffices to show that the boundary points $\pm\infty$ are inaccessible (i.e., no explosions occur). This follows easily from [**M**, p. 24] and properties of Ω_0.

II. The GHS inequality. The GHS inequality arises in the study of certain lattice models of ferromagnetism. These models consist of a finite family of real-valued random variables $\{X_i; \ i = 1, \ldots, N\}$ whose joint probability distribution on $\mathbf{R}^N$ has the form

$$\frac{1}{Z(h_1, \ldots, h_N)} \exp(-H(x_1, \ldots, x_N)) \prod_{i=1}^{N} \rho_i(dx_i).$$

H, the Hamiltonian, and Z, the partition function, are defined by

$$(2.1) \qquad H(x_1, \ldots, x_N) = -\sum_{i,j=1}^{N} J_{ij} x_i x_j - \sum_{i=1}^{N} h_i x_i,$$

$$(2.2) \qquad Z(h_1, \ldots, h_N) = \int_{\mathbf{R}^N} \exp(-H(x_1, \ldots, x_N)) \prod_{i=1}^{N} \rho_i(dx_i).$$

The indices i and j typically label atomic sites in a crystal lattice $\Lambda = \{1, \ldots, N\}$ of N sites. X_i denotes the spin of the ith atom, J_{ij} the interaction strength between X_i and X_j, and h_i the nonnegative external magnetic field strength at the ith site.

Warning. One usually requires $J_{ij} \geqslant 0$ for all $i,j = 1, \ldots, N$. In this paper, we relax this condition by allowing J_{ii} to be real for $i = 1, \ldots, N$. See Remarks 2.1 and 2.5 below.

The ρ_i are measures belonging to $\mathcal{E}$, the set of even finite measures ρ satisfying $\int \exp(kx^2)\rho(dx) < \infty$ for some $k > 0$. The choice of each ρ_i as the Bernoulli measure $(\delta(x - 1) + \delta(x + 1))/2$ defines a spin-$\frac{1}{2}$ Ising model [Th, Chapter 5]. It is assumed (if necessary) that the J_{ij}'s are sufficiently small so that the integral in (2.2) converges for all real h_i. An important thermodynamic quantity is $m(h_1, \ldots, h_N)$, the average magnetization per site, defined as

$$(2.3) \qquad m = \frac{1}{N} \sum_{i=1}^{N} E\{X_i\} = \frac{1}{N} \sum_{i=1}^{N} \frac{\partial}{\partial h_i} \ln Z(h_1, \ldots, h_N).$$

The GHS inequality states [GHS] that in a spin-$\frac{1}{2}$ model

$$\frac{\partial^3}{\partial h_i \partial h_j \partial h_k} \ln Z(h_1, \ldots, h_N) \leqslant 0 \quad \text{for all } h_i \geqslant 0,$$

$$(2.4)$$

$$J_{ii} \text{ real} \quad (i = 1, \ldots, N), \qquad J_{ij} \geqslant 0 \quad (1 \leqslant i \neq j \leqslant N),$$

and any choice of (not necessarily distinct) sites $i,j,k \in \Lambda$.

REMARK 2.1. For a spin-$\frac{1}{2}$ model, the values of the J_{ii}'s are not important because the J_{ii} terms in the Hamiltonian (2.1) contribute only the constant

$\exp(\Sigma_i J_{ii})$ to (2.2). On the other hand, our necessary and sufficient conditions in Theorem 2.4 refer to the GHS inequality in the extended form (2.4).

Notice that in its simplest form ($N = 1$, $J_{11} = 0$) (2.4) states that

$$(2.5) \qquad \frac{d^3}{dh^3} \ln \int e^{hx} \rho(dx) \leqslant 0 \quad \text{for } h \geqslant 0.$$

Applied to (2.3), the GHS inequality implies that $\partial^2 m / \partial h_j \partial h_k \leqslant 0$ so that, for example, the magnetization $m(h, \ldots, h)$ is a convex function of the external field strength h when $h \geqslant 0$, J_{ii} real, $J_{ij} \geqslant 0$ for $i \neq j$.

In [EMN] and [Sy] the GHS inequality was extended from spin-$\frac{1}{2}$ models by studying classes of non-Bernoulli measures $\rho_i \in \mathcal{S}$ for which (2.4) is valid. Inequality (2.4) is *not* true for all $\rho_i \in \mathcal{S}$. In fact, the measures

$$\rho_a(dx) = a\delta(x) + \tfrac{1}{2}(1 - a)(\delta(x - 1) + \delta(x + 1)) \quad \text{for } \tfrac{2}{3} < a < 1$$

are simple examples for which even (2.5) fails. The following theorem gives a somewhat complicated sufficient condition on the ρ_i's for (2.4) to be valid. The naturalness of this condition and its relation to $\exp(-\mathcal{V}_c)$ will become clear in the succeeding theorem.

THEOREM 2.2 [EMN]. *Given* $\rho \in \mathcal{S}$, $\mathbf{T}$ *an invertible* 4×4 *matrix, and* $F \in \mathcal{B}(\mathbf{R}^4)$ *(Borel sets in* $\mathbf{R}^4$*), let* $\rho_\mathbf{T}(F) \equiv \rho(\mathbf{T}^{-1}F)$, *where* $\rho(dx)$, $\mathbf{x} = (x^{(1)}, \ldots, x^{(4)})$, *denotes* $\Pi^4_{\alpha=1}\rho(dx^{(\alpha)})$. *We denote by* $\mathcal{G}$ *the class of all measures* $\rho \in \mathcal{S}$ *for which*

$$(2.6) \qquad \rho_\mathbf{B}(F) \geqslant \rho_\mathbf{A}(F) \quad \text{for all } F \in \mathcal{B}(\mathbf{R}^4_+),$$

where

$$\mathbf{B} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}, \quad \mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$

and $\mathbf{R}^4_+ = \{(x^{(1)}, \ldots, x^{(4)}) | x^{(\alpha)} > 0, \ \alpha = 1, \ldots, 4\}$. *Then inequality* (2.4) *holds if each* $\rho_i \in \mathcal{G}$.

REMARK 2.3. $\mathbf{A}$ and $\mathbf{B}$ are orthogonal matrices. Hence if $\rho \in \mathcal{G}$, then so is $\mu(dx) \equiv \exp(-\beta x^2)\rho(dx)$ for all β for which μ is a finite measure. We shall need this later.

It was proven in [EMN] that a measure ρ with nonvanishing C^1 density f belongs to $\mathcal{G}$ if and only if $f \in \exp(-\mathcal{V}_c)$. The next theorem considerably strengthens that result.

THEOREM 2.4. *Let* $\rho_1, \ldots, \rho_N$ *be even, finite (not identically zero) measures on* $\mathbf{R}$. *Then the following four statements are equivalent.*

(i) *for each* i, *either* $\rho_i(dx) = \text{const}(\delta(x - y) + \delta(x + y))$ *for some* $y \geqslant 0$,

or else ρ_i is absolutely continuous with respect to Lebesgue measure, and for some $I > 0$ ($I = \infty$ allowed)

$$(2.7) \qquad \frac{d\rho_i}{dx} = \begin{cases} \text{const } \exp\left(-\int_0^x G(y)\,dy\right), & x \in (-I,I), \\ 0, & x \in \mathbf{R} \setminus (-I,I), \end{cases}$$

where $G(0) = 0$ and G is convex on $[0,I)$;
  (ii) for each i, $\rho_i \in \mathcal{G}$;
  (iii) given $M \geqslant 0$ such that

$$(2.8) \quad Z(h_1, \ldots, h_N) = \int_{\mathbf{R}^N} \exp\left(\sum_{i=1}^{N} h_i x_i + \sum_{i,j=1}^{N} J_{ij} x_i x_j\right) \prod_{i=1}^{N} \rho_i(dx_i) < \infty$$

for any h_i real ($1 \leqslant i \leqslant N$), J_{ii} real, $J_{ii} < M$ ($1 \leqslant i \leqslant N$), $0 \leqslant J_{ij} \leqslant M$ ($1 \leqslant i \neq j \leqslant N$), the following inequality is valid for any $i_1, i_2, i_3 \in \{1, \ldots, N\}$ and any J_{ij} as in (2.8):

$$(2.9) \quad \frac{\partial^3}{\partial h_{i_1} \partial h_{i_2} \partial h_{i_3}} \ln Z(h_1, \ldots, h_N) \leqslant 0 \quad \text{for all } h_i \geqslant 0, i = 1, \ldots, N;$$

  (iv) for each i and all $\beta > 0$,

$$(2.10) \qquad \frac{d^3}{dh^3} \ln \int e^{hx} e^{-\beta x^2} \rho_i(dx) < 0 \quad \text{for all } h \geqslant 0.$$

REMARK 2.5. As will be seen in the proof, it was essentially known from the results of [EMN] that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) (with (iii) $\Rightarrow$ (iv) a triviality), and a weak version of (ii) $\Rightarrow$ (i) was also implicit in [EMN]. The essential new ingredient presented here is that (iv) $\Rightarrow$ (i), and the most striking corollary of this new ingredient is the equivalence of (iii) and (iv). On the other hand, the extended form of the GHS inequality given in (iii) excludes many (discrete) measures of physical interest; e.g., the measures [G]

$$(n + 1)^{-1}\{\delta(x - n) + \delta(x - n + 2) + \cdots$$
$$+ \delta(x + n - 2) + \delta(x + n)\},$$

for $n = 2, 3, \ldots$, which define spin-$n/2$ models.

REMARK 2.6. It follows from this theorem (by replacing $\rho(dx)$ by $\exp(-\beta_1 x^2)\rho(dx)$ for some $\beta_1 > 0$) that for a finite even measure ρ, the assumption, used in the definition of $\mathcal{G}$, that $\rho \in \mathcal{E}$ is redundant. From the equivalence of (ii) and (iv) and from Remark 2.3, it also follows that (2.10) need only be assumed true for all β sufficiently large and is then automatically true for all those β (including negative values) for which $\exp(-\beta x^2)\rho_i(dx)$ is a finite measure.

As a consequence of Theorem 2.4, we have the following useful facts about

$\mathcal{G}$ and $\exp(-\mathcal{V}_c)$ which will be used in the next section.

PROPOSITION 2.7. *Suppose* $f_1, f_2, \ldots, f_n, \ldots \in \exp(-\mathcal{V}_c)$. *Then*

(a) $f_1 f_2 \in \exp(-\mathcal{V}_c)$;

(b) $f_1(x)\rho(dx) \in \mathcal{G}$ *if* $\rho \in \mathcal{G}$;

(c) $g(x) \equiv \int \exp(-\beta(x-y)^2)\rho(dy) \in \exp(-\mathcal{V}_c)$ *for all* $\beta > 0$ *if and only if* $\rho \in \mathcal{G}$;

(d) *if* $f_n \to f$ *in* $L^2(\mathbf{R}; dx)$ *as* $n \to \infty$ *and* $f > 0$, *then* $f \in \exp(-\mathcal{V}_c)$.

PROOF. Part (a) follows trivially from the definition of $\mathcal{V}_c$ while part (b) follows directly from the equivalence of (i) and (ii) in Theorem 2.4. Part (c) is an immediate consequence of (ii) and (iv) in Theorem 2.4 since

$$g(x) = \exp\left(-\beta x^2 + \ln \int e^{2\beta xy} e^{-\beta y^2} \rho(dy)\right),$$

so that $(d^3/dx^3) \ln g(x) \leqslant 0$ for $x \geqslant 0$ if and only if $\rho \in \mathcal{G}$. To prove part (d), we first note that by the equivalence of (i) and (ii) in Theorem 2.4, the measures $f_n(x) \, dx$ obey inequality (2.6). Since this inequality is preserved under L^2 limits, it follows that $f(x) \, dx \in \mathcal{G}$. But since $f > 0$, f must belong to $\exp(-\mathcal{V}_c)$. $\square$

III. Proofs of Theorems 1.1, 1.2, and 1.3.

PROOF OF THEOREM 1.1. We define the function $u(t,x)$ satisfying (1.4) by the formula

$$(3.1) \qquad\qquad u(t,\cdot) = \exp(-tH_V)f,$$

where $\exp(-tH_V)$ is the continuous $L^2(\mathbf{R}; dx)$ semigroup generated by $-H_V$ [**K**, pp. 348, 491]. We first prove Theorem 1.1 for $V \equiv 0$. Indeed, denoting $\partial/\partial x$ by D, we have

$$(3.2) \quad u(t,x) = \exp\left(\frac{tD^2}{2}\right)f(x) = \frac{1}{\sqrt{2\pi t}} \int \exp\left(-(x-y)^2/2t\right)f(y) \, dy.$$

Thus the result follows from Proposition 2.7(c). Now given $V \in \mathcal{V}_c$, we have by the Trotter product formula [**C**] that

$$u(t,\cdot) = \lim_{n\to\infty} \left[\exp\left(\frac{t}{2n} D^2\right)\exp\left(-\frac{t}{n} V\right)\right]^n f,$$

so that the result follows by Proposition 2.7(a), (c) and (d) once we show that $u(t,\cdot) > 0$. We defer this until after the proof of Theorem 1.3. $\square$

REMARK 3.1. It is a consequence of the positivity of u and of Proposition 2.7(d) that $u(t,\cdot) \in C^1(\mathbf{R})$. In proving the positivity (see Lemma 3.3), we actually show that $u \in C^{1+\beta}(\mathbf{R}; C^{2+\alpha}(\mathbf{R}))$, any $0 \leqslant \alpha, \beta < 1$, and hence u is a classical solution of (1.4).

PROOF OF THEOREM 1.2. We take $f(x) = \exp(-x^2)$ and note that by the

spectral theorem

$$\Omega_0 = \frac{1}{(\Omega_0 f)} \lim_{n \to \infty} \exp(-n(H_V - E_0))f,$$

where $(\cdot,\cdot)$ denotes the $L^2(\mathbf{R};dx)$ inner product. The result now follows from Theorem 1.1, Proposition 2.7(d), and the strict positivity of Ω_0. $\quad\square$

PROOF OF THEOREM 1.3. We omit details in this proof, referring the reader to [GS] for facts about diffusion processes. For any $f \in C^2(\mathbf{R})$, we have

$$(3.3) \qquad\qquad -(H - E_0)\Omega_0 f = \Omega_0 L f$$

(where we have dropped the subscript in H_V and L_V). Since $h(t,x)$ satisfies the backward diffusion equation

$$\partial h/\partial t = Lh, \qquad h(t,\cdot) \to f \quad \text{as } t \to 0^+,$$

by (3.3)

$$h = \exp(tL)f = \Omega_0^{-1}\exp(-t(H - E_0))\Omega_0 f.$$

Part (a) of Theorem 1.3 thus follows from Theorem 1.1. For part (b), ρ_t, which satisfies a forward diffusion equation, can be written as $\rho_t = [\exp(tL)]^*\rho_0$, where $[\exp(tL)]^*$ is the adjoint semigroup of $\exp(tL)$. By (3.3)

$$\big[\exp(tL)\big]^* = \Omega_0\big[\exp(-t(H - E_0))\big]^*\Omega_0^{-1} = \Omega_0\exp(-t(H - E_0))\Omega_0^{-1},$$

so that part (b) is also a consequence of Theorem 1.1. Part (c) is immediate from Theorem 1.1 for the case $V \equiv 0$. $\quad\square$

REMARK 3.2. The potential V and the drift coefficient F in L_V are related by the Riccati equation

$$dF/dx + F^2 = 2V - 2E_0, \qquad F(0) = 0.$$

By Theorem 1.2 and equation (1.6), we see that $V \in \mathcal{V}_c$ implies F convex on $[0,\infty)$, while the result of [BL, Theorem 6.1] (quoted after Theorem 1.2) shows that V convex and in $\mathcal{V}$ implies F nondecreasing on $[0,\infty)$.

We end this section with some facts about the solution u of (1.4).

LEMMA 3.3. *The function u defined in (3.1) is strictly positive and belongs to $C^{1+\beta}(\mathbf{R};C^{2+\alpha}(\mathbf{R}))$, any $0 < \alpha,\beta < 1$.*

PROOF. For $n = 1,2,\ldots,$ take $V_n \in C^1(\mathbf{R})$ so that V_n is bounded and $\lim_{n \to \infty} V_n(x) = V(x)$ for each $x \in \mathbf{R}$. Let $u_n \in C^{1+\beta}(\mathbf{R};C^{2+\alpha}(\mathbf{R}))$, any $0 < \alpha,\beta < 1$ [F, Theorem 10, p. 72] (also $u_n(t,\cdot) \in L^2(\mathbf{R};dx)$) solve the Cauchy problem

$$\frac{\partial u_n}{\partial t} = \frac{1}{2}\frac{\partial^2 u_n}{\partial x^2} - V_n u_n, \qquad u_n(t,\cdot) \to f \quad \text{as } t \to 0^+.$$

Without loss of generality, we may assume that each V_n and V is nonnegative, and thus by the maximum principle [F, Chapter 2]

(3.4) $$0 \leqslant u_n \leqslant \sup_x f(x), \qquad n = 1,2,\ldots.$$

By [F, Theorem 15, p. 80] we conclude the existence of a function $w(t,x) \in C^{1+\beta}(\mathbf{R};C^{2+\alpha}(\mathbf{R}))$, any $0 \leqslant \alpha,\beta < 1$, such that $u_n \to w$ uniformly on compacta and which solves (1.4). Since the approximating $u_n(t,\cdot) \in L^2(\mathbf{R};dx)$, we see that $w(t,\cdot) \in L^2(\mathbf{R};dx)$, and hence $w \equiv u$. By (3.4), $u > 0$ and thus by [F, Theorem 5, p. 39] $u > 0$. □

IV. **Proof of Theorem 2.4.** We first state a lemma which will be used several times in the proof.

LEMMA 4.1 [EMN]. *Suppose that* $V: \mathbf{R} \to \mathbf{R}$ *is even and* C^1. *Then the following statements are equivalent.*
   (a) V' *is convex on* $[0,\infty)$;
   (b)

$$V(x^{(1)} + x^{(2)} + x^{(3)} + x^{(4)}) + V(x^{(1)} + x^{(2)} - x^{(3)} - x^{(4)})$$
$$+ V(x^{(1)} - x^{(2)} + x^{(3)} - x^{(4)}) + V(x^{(1)} - x^{(2)} - x^{(3)} + x^{(4)})$$
$$> V(x^{(1)} + x^{(2)} + x^{(3)} - x^{(4)}) + V(x^{(1)} - x^{(2)} + x^{(3)} + x^{(4)})$$
$$+ V(x^{(1)} + x^{(2)} - x^{(3)} + x^{(4)}) + V(x^{(1)} - x^{(2)} - x^{(3)} - x^{(4)}) \quad \text{on } \mathbf{R}^4_+;$$

   (c) $\rho(dx) \equiv \exp(-V(x))\,dx \in \mathcal{G}$.

REMARK 4.2. Part (b) can be written as

(4.1) $$\sum_{\alpha=1}^{4} V\big((\mathbf{A}^{-1}\mathbf{x})^{(\alpha)}\big) \geqslant \sum_{\alpha=1}^{4} V\big((\mathbf{B}^{-1}\mathbf{x})^{(\alpha)}\big) \quad \text{on } \mathbf{R}^4_+,$$

where $\mathbf{x} = (x^{(1)}, \ldots, x^{(4)})$ and $\mathbf{A}$ and $\mathbf{B}$ were defined in Theorem 2.2.

PROOF. We first show that (b) is equivalent to

(4.2) $$\begin{aligned}\big[V'(x^{(1)} + x^{(2)} + x^{(3)}) - V'(x^{(1)} + x^{(2)} - x^{(3)}) \big] \\ - \big[V'(x^{(1)} - x^{(2)} + x^{(3)}) - V'(x^{(1)} - x^{(2)} - x^{(3)}) \big] > 0 \quad \text{on } \mathbf{R}^3_+.\end{aligned}$$

Indeed, by the evenness of V we may rewrite (b) as

(4.3) $$\begin{aligned}x^{(4)} \int_{-1}^{1} \Big\{ \big[V'(x^{(1)} + rx^{(4)} + x^{(2)} + x^{(3)}) \\ - V'(x^{(1)} + rx^{(4)} + x^{(2)} - x^{(3)}) \big] \\ - \big[V'(x^{(1)} + rx^{(4)} - x^{(2)} + x^{(3)}) \\ - V'(x^{(1)} + rx^{(4)} - x^{(2)} - x^{(3)}) \big] \Big\}\, dr \geqslant 0 \quad \text{on } \mathbf{R}^4_+.\end{aligned}$$

Dividing by $x^{(4)}$ and letting $x^{(4)} \to 0$ shows that (4.3) implies (4.2). To see that (4.2) implies (4.3), we note that by the symmetry of V we may assume without

loss of generality that $x^{(4)} < x^{(1)}$. Now the equivalence of (a) and (b) follows from the fact that for an odd continuous V', (b) is equivalent to the convexity of V' on $[0,\infty)$. This is an elementary exercise with convex functions (see [EMN, §4] for more details). The equivalence of (b) and (c) follows from the fact that when stated in terms of V, inequality (2.6) for ρ is just (b).   □

We break the proof of Theorem 2.4 into four natural parts. The last part contains what is essentially new beyond the results of [EMN]; it is also technically the most difficult.

PROOF THAT (i) ⇒ (ii). We write ρ for ρ_i. When $\rho(dx) = \text{const}(\delta(x - y) + \delta(x + y))$, the proof that $\rho \in \mathcal{G}$ is an elementary calculation. We suppose now that ρ is absolutely continuous and first consider the case when $I = \infty$ and G in (2.7) is continuous at the origin (by its assumed convexity, $G \in C((-I,I) \setminus \{0\})$). The result then follows from the equivalence of (b) and (c) in Lemma 4.1. If G is not continuous at the origin, we may write it (as in (1.3)) as $G(x) = \tilde{G}(x) - \gamma \,\text{sgn}(x)$ with $\tilde{G}$ continuous on $\mathbf{R}$ and convex on $[0,\infty)$ and $\gamma > 0$. We then approximate G as $\lim_{n\to\infty} G_n$, where

$$G_n(x) = \begin{cases} G(x), & |x| \geqslant 1/n, \\ \tilde{G}(x) - n\gamma x, & |x| < 1/n, \end{cases}$$

and correspondingly approximate ρ by

$$(4.4) \qquad \rho_n(dx) = \text{const} \exp\left(-\int_0^x G_n(y)\,dy\right) dx.$$

Since G_n is continuous at the origin and convex on $[0,\infty)$, $\rho_n \in \mathcal{G}$. Hence $\rho \in \mathcal{G}$ because inequality (2.6) is preserved under the limit $\rho_n \to \rho$. It only remains to consider the case when $I < \infty$. We now approximate G by

$$G_n(x) = \begin{cases} G(x), & |x| \leqslant I - 1/n, \\ G(I - 1/n) + \alpha_n(x - I + 1/n), & x > I - 1/n, \\ G(-I + 1/n) + \alpha_n(x + I - 1/n), & x < -I + 1/n, \end{cases}$$

where the α_n's are chosen so that $\alpha_n \to \infty$ and $\alpha_n \geqslant D^+ G(I - 1/n)$. Again, it follows that each ρ, defined by (4.4), is in $\mathcal{G}$, and thus $\rho \in \mathcal{G}$.

PROOF THAT (ii) ⇒ (iii). For each i, we choose $\beta_i > 0$ so that $0 \leqslant J_{ii} + \beta_i$, and letting $\mu_i(x) = \exp(-\beta_i x^2)\rho_i(x)$, we rewrite (2.8) as

$$Z(h) = \int_{\mathbf{R}^N} \exp\left(\sum_{i=1}^N h_i x_i + \sum_{i,j=1}^N J'_{ij} x_i x_j\right) \prod_{i=1}^N \mu_i(dx_i),$$

where $h = (h_1, \ldots, h_N)$, each $\mu_i \in \mathcal{E}$, and each $J'_{ij} \geqslant 0$. By Remark 2.3, each $\mu_i \in \mathcal{G}$, and thus the result follows from [EMN, Theorem 1.1 and Proposition 4.1]. In order to keep the present paper self-contained, we shall give a new, independent proof, for which we need two lemmas.

LEMMA 4.3. *If $\rho \in \mathcal{G}$, then for all $m^{(1)}, \ldots, m^{(4)} = 0,1,2, \ldots$,*

(4.5)
$$0 \leqslant \int_{\mathbf{R}^4} (x^{(1)})^{m^{(1)}} \cdots (x^{(4)})^{m^{(4)}} \rho_{\mathbf{B}}(d\mathbf{x})$$

$$= \mp \int_{\mathbf{R}^4} (x^{(1)})^{m^{(1)}} \cdots (x^{(4)})^{m^{(4)}} \rho_{\mathbf{A}}(d\mathbf{x}),$$

where the minus sign holds if all the $m^{(\alpha)}$ are odd, the plus sign holds if all the $m^{(\alpha)}$ are even, and both terms are zero in all other cases.

PROOF. Because of the symmetry of ρ, both sides of (4.5) vanish unless the $m^{(\alpha)}$ are either all even or all odd [**EMN**, Theorem 2.5(c)]. In the even case, the symmetry implies that the term involving $\rho_{\mathbf{B}}$ equals the term involving $\rho_{\mathbf{A}}$, and both are positive. We now turn to the case where the $m^{(\alpha)}$ are all odd. Setting $F(x^{(1)}, \ldots, x^{(4)}) = \prod_{\alpha=1}^{4}(x^{(\alpha)})^{m^{(\alpha)}}$ and noticing that $\mathbf{B} = \Sigma \mathbf{A}$, where

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

we have

(4.6)
$$\int_{\mathbf{R}^4} F \, d\rho_{\mathbf{B}} = \int_{\mathbf{R}^4} F \, d\rho_{\Sigma \mathbf{A}} = -\int_{\mathbf{R}^4} F \, d\rho_{\mathbf{A}},$$

by oddness properties of F. This gives the equality in (4.5) (with the choice of minus sign). To show the nonnegativity in (4.5), we define $\bar{\rho} \equiv \rho_{\mathbf{B}} - \rho_{\mathbf{A}}$ and find, using (4.6), the symmetry of ρ, and oddness properties of F, that

$$2\int_{\mathbf{R}^4} F \, d\rho_{\mathbf{B}} = \int_{\mathbf{R}^4} F \, d\bar{\rho}$$

$$= \int_{\mathbf{R}^4_+} \sum_{\sigma^{(1)}, \ldots, \sigma^{(4)} = \pm 1} \sigma^{(1)} \cdots \sigma^{(4)} \, F(\sigma^{(1)} x^{(1)}, \ldots, \sigma^{(4)} x^{(4)}) \, d\bar{\rho}$$

$$= 16 \int_{\mathbf{R}^4_+} F \, d\bar{\rho}.$$

But this is positive since $\rho \in \mathcal{G}$ implies that $\bar{\rho}$ is a positive measure on $\mathbf{R}^4_+$. $\square$

Given $h = (h_1, \ldots, h_N) \in \mathbf{R}^N$, we write $h \geqslant 0$ if each $h_i \geqslant 0$. For $h^{(1)}, \ldots, h^{(4)} \in \mathbf{R}^N$, $\mathbf{T}$ an invertible 4×4 matrix, we let $\mathbf{h}_i = (h_i^{(1)}, \ldots, h_i^{(4)}) \in \mathbf{R}^4$, $i = 1, \ldots, N$, and define

$$h_{\mathbf{T}}^{(\theta)} = \left((\mathbf{T}^{-1}\mathbf{h}_1)^{(\theta)}, \ldots, (\mathbf{T}^{-1}\mathbf{h}_N)^{(\theta)} \right) \in \mathbf{R}^N, \qquad \theta = 1, \ldots, 4.$$

LEMMA 4.4. *Let $h \to f(h)$ be an even real-valued function in $C^3(\mathbf{R}^N)$. Then*

$$\frac{\partial^3 f(h)}{\partial h_i \partial h_j \partial h_k} > 0 \quad \text{for all } i,j,k = 1, \ldots, N, \text{ all } h > 0,$$

if

(4.7) $$\sum_{\alpha=1}^{4} f(h_A^{(\alpha)}) > \sum_{\alpha=1}^{4} f(h_B^{(\alpha)}) \quad \text{for all } h^{(1)}, \ldots, h^{(4)} > 0.$$

We omit the proof as it is similar to that of Lemma 4.1.

We return to the proof that (ii) $\Rightarrow$ (iii) in Theorem 2.4. By the restriction on the J_{ij}'s, we note that $Z(h)$ is an entire function of its arguments. Hence, by Lemma 4.4, we must show

(4.8) $$\overline{Z}(h^{(1)}, \ldots, h^{(4)}) \equiv \prod_{\alpha=1}^{4} Z(h_B^{(\alpha)}) - \prod_{\alpha=1}^{4} Z(h_A^{(\alpha)}) > 0$$

$$\text{for all } h^{(1)}, \ldots, h^{(4)} \geq 0.$$

Given $x^{(1)}, \ldots, x^{(4)} \in \mathbf{R}^N$ and setting $\mathbf{x}_i = (x_i^{(1)}, \ldots, x_i^{(4)}) \in \mathbf{R}^4$, we can rewrite $\overline{Z}$ as

$$\overline{Z} = \int_{\mathbf{R}^{4N}} \left[\exp\left(\sum_{i=1}^{N} \langle \mathbf{B}^{-1}\mathbf{h}_i, \mathbf{x}_i \rangle \right) - \exp\left(\sum_{i=1}^{N} \langle \mathbf{A}^{-1}\mathbf{h}_i, \mathbf{x}_i \rangle \right) \right]$$

$$\times \exp\left(\sum_{i,j=1}^{N} J'_{ij}\langle \mathbf{x}_i, \mathbf{x}_j \rangle \right) \prod_{i=1}^{N} \mu_i(d\mathbf{x}_i),$$

where $\langle \cdot, \cdot \rangle$ denotes the $\mathbf{R}^4$ inner product. Setting

$$\tau(d\mathbf{x}_1, \ldots, d\mathbf{x}_N) = \prod_{i=1}^{N} (\mu_i)_{\mathbf{B}}(d\mathbf{x}_i) - \prod_{i=1}^{N} (\mu_i)_{\mathbf{A}}(d\mathbf{x}_i)$$

and using the fact that $\mathbf{A}$ and $\mathbf{B}$ are orthogonal matrices, we have

$$\overline{Z} = \int_{\mathbf{R}^{4N}} \left[\exp\left(\sum_{i=1}^{N} \langle \mathbf{h}_i, \mathbf{B}\mathbf{x}_i \rangle \right) - \exp\left(\sum_{i=1}^{N} \langle \mathbf{h}_i, \mathbf{A}\mathbf{x}_i \rangle \right) \right]$$

$$\times \exp\left(\sum_{i,j=1}^{N} J'_{ij}\langle \mathbf{x}_i, \mathbf{x}_j \rangle \right) \prod_{i=1}^{N} \mu_i(d\mathbf{x}_i)$$

$$= \int_{\mathbf{R}^{4N}} \exp\left(\sum_{i=1}^{N} \langle \mathbf{h}_i, \mathbf{x}_i \rangle + \sum_{i,j=1}^{N} J'_{ij}\langle \mathbf{x}_i, \mathbf{x}_j \rangle \right) \tau(d\mathbf{x}_1, \ldots, d\mathbf{x}_N).$$

Now (4.8) follows if we can show the nonnegativity of all the multi-Taylor coefficients of $\overline{Z}$ at the origin (since $\overline{Z}$ is an entire function of the $h_i^{(\alpha)}, J'_{ij}$). Since each $h_i^{(\alpha)}$ and J'_{ij} are nonnegative, it suffices to prove

$$\int_{\mathbf{R}^{4N}} \prod_{i=1}^{N} \left(x_i^{(1)}\right)^{m_i^{(1)}} \cdots \left(x_i^{(4)}\right)^{m_i^{(4)}} \tau(d\mathbf{x}_1, \ldots, d\mathbf{x}_N) > 0$$

for all $m_i^{(\alpha)} = 0, 1, 2, \ldots$.

But this follows from Lemma 4.3. $\square$

PROOF THAT (iii) $\Rightarrow$ (iv). We choose $J_{11} = -\beta$, $J_{ij} = 0$ for all $2 < i,j < N$, $h_1 = h$, $h_i = 0$ for $i = 2, \ldots, N$, and $i_1 = i_2 = i_3 = 1$. Then inequality (2.9) becomes identical to inequality (2.10).

PROOF THAT (iv) $\Rightarrow$ (i). Writing ρ for ρ_i, we express ρ as the weak limit of ρ_β as $\beta \to \infty$, where

$$\frac{d\rho_\beta}{dx} = \int_{\mathbf{R}} \sqrt{\beta/\pi} \, e^{-\beta(x-y)^2} \rho(dy)$$

(i.e., $\int f \, d\rho = \lim_{\beta\to\infty} \int f \, d\rho_\beta$ for any bounded continuous f). As in the proof of Proposition 2.7(c), it follows from (2.10) that

$$\frac{d\rho_\beta}{dx} = \exp\left[-V_\beta(0) - \int_0^x G_\beta(y) \, dy \right]$$

with $G_\beta(0) = 0$, G_β smooth (in fact, real analytic), and G_β convex on $[0,\infty)$ (i.e., $d^2 G_\beta/dx^2 > 0$ for $x > 0$). It thus suffices to show that the only finite measures obtainable as weak limits of such ρ_β's are those given in Theorem 2.4(i). This will be done after a somewhat long-winded series of elementary lemmas.

LEMMA 4.5. *Define* $y_\beta = \sup\{x \mid x > 0, \, G_\beta(x) = 0\}$. *Then* $\limsup_{\beta\to\infty} y_\beta < \infty$.

PROOF. Since $G_\beta(0) = 0$ and G_β is convex on $[0,\infty)$, it follows that on $[0, y_\beta]$, $G_\beta < 0$ and $\exp(-V_\beta)$ is nondecreasing. Then for all $x, \varepsilon > 0$ with $x + \varepsilon < y_\beta$,

(4.9) $\rho_\beta([\varepsilon, x + \varepsilon]) > \frac{1}{2} \rho_\beta([-x, x])$.

If $\limsup y_\beta = \infty$, then by choosing an appropriate subsequence of β's and taking the limit of (4.9), we would have for all $x, \varepsilon > 0$ that $\rho([\varepsilon, x + \varepsilon]) > \frac{1}{2} \rho((-x, x))$. But this is impossible for a finite measure ρ. $\square$

By choosing a subsequence of β's (if necessary), we may now assume without loss of generality that $y_\infty = \lim_{\beta\to\infty} y_\beta$ exists and is nonnegative.

LEMMA 4.6. *If for some* $x_1 > 0$, $\limsup_{\beta\to\infty} G_\beta(x_1) = \infty$, *then* $\rho((x_1,\infty)) = 0$.

PROOF. If $G_\beta(x_1) > 0$, then on $[x_1,\infty)$, $G_\beta' > 0$ and G_β is nondecreasing. Thus for $x_1 < x_2$ and $\xi > 0$,

(4.10) $\begin{aligned} V_\beta(x_2 + \xi) &> V_\beta(x_1 + \xi) + [V_\beta'(x_1 + \xi)](x_2 - x_1) \\ &> V_\beta(x_1 + \xi) + V_\beta'(x_1) \cdot (x_2 - x_1). \end{aligned}$

Exponentiating (4.10) and integrating the resulting inequality yields that

(4.11) $\rho_\beta([x_2,\infty)) \leqslant \exp(-G_\beta(x_1) \cdot (x_2 - x_1))\rho_\beta([x_1,\infty)).$

Thus, if $\limsup G_\beta(x_1) = \infty$, then $\rho((x_2,\infty)) = 0$ for all $x_2 > x_1$, which implies that $\rho((x_1,\infty)) = 0$, as desired. $\square$

LEMMA 4.7. *Define* $K_\beta = -\inf\{G_\beta(x)|x > 0\}$. *Then if* $K_\beta > 0$ (*so that* $y_\beta > 0$),

(4.12) $G_\beta(x) \leqslant -K_\beta/2 + K_\beta|x - y_\beta/2|/y_\beta \equiv F_\beta(x), \quad x \in [0,y_\beta],$

(4.13) $G_\beta(x) \geqslant K_\beta(x - y_\beta)/y_\beta, \quad x \in [y_\beta,\infty).$

PROOF. Pick $w > 0$ so that $G_\beta(w) = -K_\beta$. Then $0 < w < y_\beta$, and using the convexity of G_β and the fact that $G_\beta(0) = 0$, we have

(4.14) $G_\beta(x) \leqslant -K_\beta x/w \leqslant -K_\beta x/y_\beta, \quad x \in [0,w],$

(4.15) $G_\beta(x) \leqslant K_\beta(x - y_\beta)/(y_\beta - w) \leqslant K_\beta(x - y_\beta)/y_\beta,$
$$x \in [w, y_\beta],$$

(4.16) $G_\beta(x) \geqslant G_\beta'(y_\beta) \cdot (x - y_\beta) \geqslant K_\beta(x - y_\beta)/(y_\beta - w),$
$$x \in [y_\beta, \infty).$$

Now (4.16) implies (4.13) while (4.12) follows from (4.14) and (4.15) since $F_\beta(x)$ in (4.12) is just $\max\{-K_\beta x/y_\beta, K_\beta(x - y_\beta)/y_\beta\}$. $\square$

LEMMA 4.8. *If* $\limsup K_\beta = \infty$, *then*
$$\rho(x) = (\delta(x - y) + \delta(x + y))/2 \quad \text{with } y = y_\infty.$$

PROOF. If $\limsup K_\beta = \infty$, then by (4.13) $\limsup G_\beta(x) = \infty$ for any $x > y_\infty$. Hence by Lemma 4.6 $\rho((y_\infty,\infty)) = 0$. If $y_\infty = 0$, then $\rho(x) = \delta(x)$, and we are finished. In order to complete the proof, we suppose $y_\infty > 0$ and proceed to show that $\rho((-y_\infty, y_\infty)) = 0$. To accomplish this, we use (4.12) to derive that for $0 \leqslant x_1 \leqslant x_2 \leqslant y_\beta$,

$$V_\beta(x_1) = V_\beta(x_2) - \int_{x_1}^{x_2} G_\beta(y)\,dy \geqslant V_\beta(x_2) - \int_{x_1}^{x_2} F_\beta(y)\,dy$$

(4.17) $\geqslant V_\beta(x_2) - \int_0^{x_{12}} F_\beta(y)\,dy - \int_{y_\beta - x_{12}}^{y_\beta} F_\beta(y)\,dy$

$$= V_\beta(x_2) + K_\beta(x_2 - x_1)^2/4y_\beta,$$

where $x_{12} \equiv (x_2 - x_1)/2$. Exponentiating (4.17) and integrating the resulting inequality leads to the fact that for $\mu, \varepsilon > 0$ with $\mu + \varepsilon < y_\beta$,

$$\rho_\beta([-\mu,\mu]) \leqslant 2\exp(-K_\beta \varepsilon^2/4y_\beta)\rho_\beta([\varepsilon, \mu + \varepsilon]).$$

Hence, if lim sup $K_\beta = \infty$, it follows that for $\mu + \varepsilon < y_\infty$, $\rho((-\mu,\mu)) = 0$. Letting $\varepsilon \to 0$ gives $\rho((-y_\infty, y_\infty)) = 0$, as desired. □

LEMMA 4.9. *If* lim $\sup_{\beta \to \infty} K_\beta < \infty$, *then* $\rho(\{x\}) = 0$ *for every* $x \neq 0$.

PROOF. For $0 < x_1 < x_2$,

$$V_\beta(x_2) = V_\beta(x_1) + \int_{x_1}^{x_2} G_\beta(y)\,dy \geq V_\beta(x_1) - K_\beta(x_2 - x_1),$$

so that for $0 < \varepsilon < x_1 < x_2$,

(4.18) $\rho_\beta([x_2 - \varepsilon, x_2 + \varepsilon]) \leq \exp(K_\beta(x_2 - x_1))\rho_\beta([x_1 - \varepsilon, x_1 + \varepsilon])$.

Thus, letting $\beta \to \infty$, we have

$$\begin{aligned}
(4.19) \quad \rho(\{x_2\}) &\leq \rho((x_2 - \varepsilon, x_2 + \varepsilon)) \\
&\leq \exp(K_\infty(x_2 - x_1))\rho([x_1 - \varepsilon, x_1 + \varepsilon]),
\end{aligned}$$

where $K_\infty = \lim \sup K_\beta$. Given $x_2 > 0$, we pick $x_1 \in (0, x_2)$ such that $\rho(\{x_1\}) = 0$ and let $\varepsilon \to 0$ in (4.19) to show that $\rho(\{x_2\}) = 0$. □

LEMMA 4.10. *If* lim $\sup_{\beta \to \infty} K_\beta < \infty$ *and for some* $\bar{x} > 0$, lim $\sup_{\beta \to \infty} G_\beta(\bar{x}) < \infty$, *then* $\rho(x)$ *is absolutely continuous on* $(-\bar{x}, \bar{x})$ *and*

$$\frac{d\rho}{dx} = \text{const} \exp\left(-\int_0^x G(y)\,dy\right), \quad x \in (-\bar{x}, \bar{x}),$$

where $G(0) = 0$ *and* G *is convex on* $[0, \bar{x})$.

PROOF. Under the assumptions of the lemma, there exists a constant

$$K = \lim_{\beta \to \infty} \sup\left(\max\{K_\beta, G_\beta(\bar{x})\}\right) < \infty$$

such that

(4.20) $$|G_\beta(x)| \equiv |V_\beta'(x)| \leq K, \quad x \in [-\bar{x}, \bar{x}],$$

and thus we have that

(4.21) $\quad V_\beta(0) - K\bar{x} \leq V_\beta(x) \leq V_\beta(0) + K\bar{x}, \quad x \in [-\bar{x}, \bar{x}]$.

If lim inf $V_\beta(0) = -\infty$, then by the second inequality in equation (4.21) lim sup $\rho_\beta((-\bar{x}, \bar{x})) = \infty$. If lim sup $V_\beta(0) = \infty$, then by the first inequality in (4.21), lim inf $\rho_\beta((-\bar{x}, \bar{x})) = 0$, which by (4.18) would imply that

$$\lim \inf \rho_\beta([x_2 - \varepsilon, x_2 + \varepsilon]) = 0 \quad \text{for all } 0 < \varepsilon < x_2.$$

Since ρ is finite and not identically zero, neither of these two cases is possible, and consequently $V_\beta(0)$ is bounded above and below as $\beta \to \infty$. By choosing a subsequence of β's (if necessary), we may assume without loss of generality that $\lim_{\beta \to \infty} V_\beta(0) \equiv V_\infty$ exists. Since the G_β's are convex and uniformly bounded on $[0, \bar{x}]$ (by (4.20)), we may assume, again without loss of generality

(by choosing a subsequence, if necessary), that there exists a convex function G on $[0,\bar{x}]$ such that $G(0) = 0$ and $G_\beta \to G$ pointwise on $[0,\bar{x}]$. Since G must be continuous on $(0,\bar{x})$, it follows that $G_\beta \to G$ uniformly on compact subsets of $(0,\bar{x})$. After extending G to $[-\bar{x},\bar{x}]$ by oddness and using (4.20), we have that

$$V_\beta(x) = V_\beta(0) + \int_0^x G_\beta(y)\, dy \to V_\infty + \int_0^x G(y)\, dy \quad \text{as } \beta \to \infty$$

uniformly on $[-\bar{x},\bar{x}]$. Hence

$$\rho(dx) = \lim_{\beta \to \infty} \exp(-V_\beta(x))\, dx$$

has the stated form. □

PROOF THAT (iv) ⇒ (i) COMPLETED. By Lemma 4.8, it suffices to consider the case when $\limsup K_\beta < \infty$. If for all $\bar{x} > 0$, $\limsup G_\beta(\bar{x}) < \infty$, then by Lemma 4.10 ρ has the form given in (i) with $I = \infty$. On the other hand, if for arbitrarily small $\bar{x} > 0$, $\limsup G_\beta(\bar{x}) = \infty$, then by Lemma 4.6, $\rho(x) = \delta(x)$, and we are finished. Consequently we may define

$$x_0 = \inf\left\{ \bar{x} \,\Big|\, \bar{x} > 0,\ \limsup_{\beta \to \infty} G_\beta(\bar{x}) = \infty \right\}$$

and assume without loss of generality that $0 < x_0 < \infty$. It then follows that $\rho((x_0,\infty)) = 0$ by Lemma 4.6, that $\rho(\{x_0\}) = 0$ by Lemma 4.9, and that, for any $\varepsilon > 0$, ρ on $(-(x_0 - \varepsilon), x_0 - \varepsilon)$ is absolutely continuous with the form given by Lemma 4.10. Letting $\varepsilon \to 0$, we obtain the form of (i) with $I = x_0$. □

REFERENCES

[BJS] L. Bers, F. John and M. Schechter, *Partial differential equations*, Interscience, New York, 1964. MR **29** #346.

[BL] H. J. Brascamp and E. H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions and with an application to the diffusion equation*, J. Functional Anal. **22** (1976), 366–389.

[C] P. R. Chernoff, *Note on product formulas for operator semigroups*, J. Functional Analysis **2** (1968), 238–242. MR **37** #6793.

[EMN] R. S. Ellis, J. L. Monroe and C. M. Newman, *The GHS and other correlation inequalities for a class of even ferromagnets*, Comm. Math. Phys. **46** (1976), 167–182.

[F] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs, N. J., 1964. MR **31** #6062.

[GS] I. Gihman and A. Skorohod, *Introduction to the theory of random processes*, "Nauka", Moscow, 1965; English transl., Saunders, Philadelphia, Pa., 1969. MR **33** #6689; **40** #923.

[G] R. B. Griffiths, *Rigorous results for Ising ferromagnets of arbitrary spin*, J. Mathematical Phys. **10** (1969), 1559–1565. MR **41** #1338.

[GHS] R. B. Griffiths, C. A. Hurst and S. Sherman, *Concavity of magnetization of an Ising ferromagnet in a positive external field*, J. Mathematical Phys. **11** (1970), 790–795. MR **42** #1412.

[K] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1966. MR **34** #3324.

[M] P. Mandl, *Analytical treatment of one-dimensional Markov processes*, Academia, Prague;

Springer-Verlag, Berlin and New York, 1968. MR **40** #930.

[S1] B. Simon, *Coupling constant analyticity for the anharmonic oscillator*, Ann. Physics **58** (1970), 76–136.

[S2] _____, *The* $P(\varphi)_2$ *Euclidean (quantum) field theory*, Princeton Univ. Press, Princeton, N. J., 1974.

[S3] _____, *Approximation of Feynman integrals and Markov fields by spin systems*, Proc. Internat. Congress Math., Vol. 2 (Vancouver, B. C., 1974), Canad. Math. Congress, 1975, pp. 399–402.

[Sy] G. Sylvester, *Continuous spin inequalities for Ising ferromagnets*, Harvard/M.I.T., 1975 (preprint).

[Th] C. Thompson, *Mathematical statistical mechanics*, Macmillan, New York, 1972.

[Ti] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, Clarendon Press, Oxford, 1946. MR **8**, 458.

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